Exam 10 Review

Solution Guide with Metacognitive Scaffolding

BC Precalculus | LASA

Sequences • Arithmetic / Geometric • Convergence • Power & Taylor Series

The opening pages of a real session deliverable from Engineering Confidence, prepared for a real student ahead of a real exam — only the name has been removed. The family received the complete 26-page guide.

Before You Compute

About this review. The review packet has 21 problems in flashcard format — each page shows the previous answer above the next problem, so the topics jump around. That's great for studying, because it forces you to do the hardest step first — *tool selection* — on every problem.

This guide presents the 21 problems in the **same order as the printed review packet**, so you can work alongside it. Each header names the concept the problem tests. The next two pages are your **concept reference**: a diagnostic decision tree and a Master Toolbox. Read those first, then work through the problems and refer back as needed.

Diagnostic Decision Tree

How to read the prompt

When you sit down with a problem, run through these questions *in order*. Stop at the first "yes."

1. Is it a sequence? (one term, or the formula for a_n)

- Look at numerators, denominators, and signs separately.
- Test your formula by plugging in $n = 1, 2, 3$ and matching given terms.
- Tools: arithmetic ($a_n = a_1 + (n-1)d$), geometric ($a_n = a_1 r^{n-1}$), or alternating signs via $(-1)^n$ or $(-1)^{n+1}$.

2. Is it a finite sum (something like $1 + 4 + 7 + \cdots + 295$, or $\sum_{n=1}^{8} \cdots$)?

- Constant difference between terms? **Arithmetic.** Use $S_n = \frac{n}{2}(a_1 + a_n)$.
- Constant ratio between terms? **Geometric.** Use $S_n = a_1 \cdot \dfrac{1 - r^n}{1 - r}$.

3. Is it an infinite sum (upper limit is ∞)? Run the convergence pipeline:

(a) **Geometric with $|r| < 1$?** Use $S = \dfrac{a_1}{1 - r}$.
(b) **Test for Divergence:** compute $\lim_{n\to\infty} a_n$. If it isn't 0, write *diverges* and stop.
(c) **Direct Comparison:** bound your terms above by a known convergent series, or below by a known divergent one (often the harmonic $\sum 1/n$).

4. Is it a repeating decimal? Express the repeating tail as an infinite geometric series. First repeated block at its actual decimal location $= a_1$; ratio $= 10^{-k}$ where k is the block length. Apply $S = \frac{a_1}{1-r}$, then add the non-repeating prefix.

5. Is there an x inside the series? ("Find the interval of convergence.") Treat the bracketed x-expression as the ratio of a geometric series. Set $|\text{bracket}| < 1$ and solve for x.

6. Does the series have factorials in the denominator? Probably a Taylor series. Match against the three you should have memorized:

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
- $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

Master Toolbox — All the Formulas You Need

Sequences

Arithmetic: common *difference* d.

- nth term: $a_n = a_1 + (n-1)d$
- Recursive form: $a_n = a_{n-1} + d$

Geometric: common *ratio* r.

- nth term: $a_n = a_1 \cdot r^{n-1}$
- Recursive form: $a_n = r \cdot a_{n-1}$

Identifying which one: subtract consecutive terms (constant $\Rightarrow$ arithmetic) *or* divide consecutive terms (constant $\Rightarrow$ geometric).

Series sums

Arithmetic: $S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}[2a_1 + (n-1)d]$

Geometric (finite): $S_n = a_1 \cdot \frac{1-r^n}{1-r}$ (use whenever $r \neq 1$)

Geometric (infinite): $S = \frac{a_1}{1-r}$ (only if $|r| < 1$ — otherwise the sum diverges)

Sigma-notation arithmetic: $\sum_{n=1}^{N} c = cN, \quad \sum_{n=1}^{N} n = \frac{N(N+1)}{2}$

Convergence tests for $\sum_{n=1}^{\infty} a_n$

1. **Geometric series test.** If $a_n = a_1 r^{n-1}$, the series converges iff $|r| < 1$.
2. **Test for Divergence (TfD).** If $\lim_{n\to\infty} a_n \neq 0$ (including doesn't exist), the series diverges. *If the limit is 0, the test is inconclusive — it does **not** prove convergence.*
3. **Direct Comparison.** Suppose $0 \leq a_n \leq b_n$ for all n past some point.
   - If $\sum b_n$ **converges**, then $\sum a_n$ converges (smaller than something finite).
   - If $\sum a_n$ (the smaller one) **diverges**, then $\sum b_n$ also diverges.

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges — this is your reference for "terms going to 0 but the sum still blowing up."

Limit-of-a_n shortcuts

You'll use these inside the Test for Divergence and inside Taylor recognition:

- **Rational** $a_n = \dfrac{P(n)}{Q(n)}$**:** compare degrees of top and bottom.
  - Top deg $<$ bottom deg $\Rightarrow$ limit $= 0$.
  - Top deg $=$ bottom deg $\Rightarrow$ limit $=$ ratio of leading coefficients.
  - Top deg $>$ bottom deg $\Rightarrow$ limit $= \pm\infty$.
- **Geometric** r^n**:** $\to 0$ if $|r| < 1$; $\to +\infty$ if $r > 1$; no limit (oscillates with growing magnitude) if $r < -1$; oscillates if $r = -1$; $\to 1$ if $r = 1$.
- **"Drop the small terms."** For limits at infinity, only the dominant term in numerator and denominator matters. The $+5$ in $4n+5$, the $+21$ in $73n+21$, and the $(-1)^n$ in $37n+(-1)^n$ are all bounded $\Rightarrow$ ignore them when computing the limit.

Taylor series — memorize these three

For all real x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Pattern recognition cues:

- Factorial $n!$ in denominator and x^n on top $\Rightarrow e^x$.
- Alternating sign, factorial *odd* $(2n+1)!$, odd power $x^{2n+1} \Rightarrow \sin x$.
- Alternating sign, factorial *even* $(2n)!$, even power $x^{2n} \Rightarrow \cos x$.

Bonus geometric: for $|x| < 1$, $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. (This is just the infinite-geometric-series formula written as a function of x.)

The 21 Problems — In Review-Packet Order

The problems appear below in the same order they're given in the printed review packet. Each header notes the concept the problem tests — the Master Toolbox on the previous pages is your concept reference. Work the problem first, then read the scaffolding.

Problem 1 (P1): repeating decimal with non-repeating prefix

Use a geometric series to write $-5.8\overline{67}$ *as a ratio of integers.*

Before You Compute

Reading the decimal. The bar is over "67" only, so $-5.8\overline{67} = -5.86767676\ldots$. The "8" is part of a fixed (non-repeating) prefix; the "67" is the repeating block.

The question explicitly says "use a geometric series." So our job is to express the repeating tail as $\sum_{n=0}^{\infty} a_1 r^n$ and apply the infinite-geometric-sum formula $S = \dfrac{a_1}{1-r}$.

Strategy.

1. Pull the negative sign out front; deal with $5.8\overline{67}$ as a positive number, then re-apply the sign at the end.
2. Split into non-repeating + repeating: $5.8\overline{67} = 5.8 + 0.0\overline{67}$.
3. Write the repeating tail $0.0\overline{67}$ as an infinite geometric series in sigma notation.
4. Identify a_1 and r, check $|r| < 1$, apply $S = \dfrac{a_1}{1-r}$.
5. Add to 5.8, re-apply the negative.

Step 1: Split the decimal. The non-repeating part is $5.8 = \frac{58}{10}$, and the repeating tail is $0.0\overline{67} = 0.0676767\ldots$:

$$5.8\overline{67} \;=\; 5.8 \;+\; 0.0\overline{67}.$$

Step 2: Write the repeating tail as a sum of fractions. The first "67" lives in positions 10^{-2} and 10^{-3}:

$$0.067 = \frac{67}{1000} = \frac{67}{10^3}.$$

The next "67" is shifted two more decimal places to the right (since the block has length 2), so it contributes $\dfrac{67}{10^5}$. Then $\dfrac{67}{10^7}$, then $\dfrac{67}{10^9}$, and so on:

$$0.0\overline{67} \;=\; \frac{67}{10^3} + \frac{67}{10^5} + \frac{67}{10^7} + \frac{67}{10^9} + \cdots$$

Step 3: Put in sigma notation and identify the geometric pieces. Pull out the common factor $\frac{67}{10^3}$:

$$0.0\overline{67} \;=\; \frac{67}{10^3}\left(1 \;+\; \frac{1}{10^2} \;+\; \frac{1}{10^4} \;+\; \frac{1}{10^6} \;+\; \cdots\right) \;=\; \frac{67}{1000}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n.$$

This is an infinite geometric series with

$$a_1 \;=\; \frac{67}{1000}, \qquad r \;=\; \frac{1}{100}.$$

Convergence check: $|r| = \frac{1}{100} < 1$ ✓, so the sum exists.

Step 4: Apply $S = \dfrac{a_1}{1-r}$.

$$0.0\overline{67} \;=\; \frac{67/1000}{1-1/100} \;=\; \frac{67/1000}{99/100} \;=\; \frac{67}{1000}\cdot\frac{100}{99} \;=\; \frac{67}{990}.$$

Step 5: Combine with the non-repeating part. Use the common denominator 990:

$$5.8 \;=\; \frac{58}{10} \;=\; \frac{58\cdot 99}{990} \;=\; \frac{5742}{990}.$$

$$5.8\overline{67} = \frac{5742}{990} + \frac{67}{990} = \frac{5809}{990}.$$

Step 6: Re-apply the negative sign.

$$-5.8\overline{67} = -\frac{5809}{990}.$$

Sanity check (decimals): $-5809/990 \approx -5.867676\ldots$ ✓

Answer: $-\dfrac{5809}{990}$

Watch Out

Three places to slip:

- **Finding a_1.** It's the value of the *first* repeated piece in its actual decimal location, not just the digits "67." Here that first piece is $0.067 = 67/1000$, not 67 on its own.
- **Finding r.** It's the multiplier that takes you from one block to the next. A block of length k shifts you k decimal places to the right each time, so $r = 10^{-k}$. Here the block "67" has length 2, so $r = 1/100$.
- **Handling the non-repeating prefix.** Don't try to bake 5.8 into the geometric series. Just compute the repeating tail as a series, then *add* the non-repeating part at the end. (Common denominator 990 makes the final fraction clean.)

Connection

Faster alternative (the shift-and-subtract trick). Once you trust the geometric-series result, here's a quicker mechanical method for future repeating-decimal problems. Set $x = -5.8\overline{67}$, then:

$$10x = -58.\overline{67}, \qquad 1000x = -5867.\overline{67}.$$

Subtracting: $1000x - 10x = -5867 - (-58) = -5809$, so $990x = -5809$ and $x = -\dfrac{5809}{990}$.

The two methods give the same answer because they're the same math: $990 = 10^{j+k} - 10^j$ where j is the number of non-repeating decimal digits between the decimal point and the repeating block (here $j = 1$, the "8") and k is the block length. The shift-and-subtract trick is just a streamlined way to compute the geometric sum without writing the series.

On the test: since the wording says "use a geometric series," show the series setup explicitly — $a_1 = 67/1000$, $r = 1/100$, $S = a_1/(1-r) = 67/990$ — and then combine with the non-repeating part. Don't lead with shift-and-subtract on this one.

The guide continues for another twenty pages — all 21 review problems, each worked to this standard, through the answer key and exam-day reminders.

Every family receives the complete packet, every session.