ENGINEERING CONFIDENCE

Limits & Continuity

Solution Guide with Metacognitive Scaffolding

AP CALCULUS AB · UNIT 1

Evaluating limits · The squeeze theorem · Limits at infinity · Continuity · The Intermediate Value Theorem

A reference for the whole unit — not a single session's homework, but built the way my session guides are built: tool selection first, every step shown, commentary where the thinking usually goes wrong.

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BEFORE YOU COMPUTE

About this guide. One sentence runs under every question in this unit: *a limit is about the approach, never the arrival*. Where the function was *heading* as x closes in on a, and what it actually does *at* a, are two separate facts — and Unit 1 is built almost entirely out of the cases where they disagree. That is why so much of it feels like a trick until you see that it never was one.

The habit that turns that sentence into procedure is *diagnose before you compute*. Substitution is the stethoscope: what it returns tells you which tool the problem actually needs, and students who skip that diagnosis end up doing algebra on autopilot.

Three things go wrong in this unit, and they go wrong for nearly everybody. This guide goes after each one deliberately rather than hoping you absorb it in passing.

- **Answering the wrong question.** Reporting where the function *landed* when the question asked where it was *heading*. Every other error below is a variation on this one.
- **Treating $\frac{0}{0}$ as a verdict.** It is not an answer and it is not a failure. It is an instruction. Reading it as "undefined" and stopping is the most common way to lose a problem you were fully able to finish.
- **Writing an answer where a justification was asked for.** On the continuity and Intermediate Value Theorem questions, the exam grades a *sentence*. A correct number with the reasoning left in your head scores less than the same number with the reasoning said out loud.

The pages that follow are your **concept reference**: a diagnostic decision tree and a Master Toolbox. Read those first. Then work each problem *before* reading its solution — the "Before you compute" notes are there to catch you at the exact moment a wrong turn usually happens.

What is and isn't in here. Unit 1 is sixteen topics, and the reference sections ahead carry all sixteen except the first — "can change occur at an instant?" is the question the whole course is an answer to, and Unit 2 is where it gets one. Everything else has a home here: the limit laws, evaluating limits every way they are asked, one-sided limits, the squeeze theorem and the trigonometric limits, end behavior, continuity at a point and across an interval, the types of discontinuity, and the Intermediate Value Theorem.

What ten problems *cannot* do is give you enough repetitions. Each one is chosen to be the clearest instance of its shape rather than one of many, so treat a problem you found easy as a signal to go find five more like it, not as a topic finished. This guide is built to make you fast at recognizing which question you are looking at. Fluency is volume, and volume is homework.

Diagnostic Decision Tree

HOW TO READ THE PROMPT

1. **Given a graph or a table?** → Read, don't compute. Trace each side separately, then compare them only at the end.

2. **Piecewise, or an unknown constant?** → The seam is the whole problem. A one-sided limit from each piece's own formula, then compare — set them equal to pin a constant. Don't substitute here: at a seam it answers with one piece and ignores the other.

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above: what am I holding? — below: what do I do with it?

3. **Substitute $x = a$. What came back?** →
   a number — that is the limit. **Stop.**
   nonzero$/0$ — vertical asymptote. Settle the sign on each side.
   $0/0$ — indeterminate. Go to **4**.

4. **It is $0/0$ — which move?** →
   polynomials — factor and cancel
   a square root — multiply by the conjugate
   a fraction inside a fraction — combine it first
   a sine or a cosine — a special limit, not algebra
   then substitute again.

5. **Is x heading to $\pm\infty$?** → End behavior. Circle the dominant term top and bottom. Check the sign if a root meets $x \to -\infty$.

6. **About continuity?** → Two stages. The three-part checklist settles *whether*; then what the limit was doing settles *which kind*.

7. **Show that a value exists?** → IVT. Continuous *and why*; both endpoint values; then cite the theorem.

Rungs 1 and 2 are not tools. They ask what kind of object you are holding, and they come first because substitution cannot read a graph, a table, or a seam. From rung 3 down, stop at the first "yes" — each rung rules out a whole family of methods, so you never have to choose between seven techniques at once.

Four of those rungs hide a reason worth carrying, because a reason is what lets you rebuild the step on a morning you have forgotten it.

Why the first two come before any arithmetic. They are the only rungs that ask what you are *holding*, and a graph, a table, and a piecewise rule are three objects substitution cannot read. The first two have no formula to substitute into. The third has *two* formulas, and substitution will quietly pick one of them and answer as though the other did not exist — Problem 5(c) is that trap drawn out, a function whose value at the seam is a perfectly ordinary number while the limit does not exist at all. Settle what you are looking at, and rung 3 becomes trustworthy.

Why substitution comes next rather than last. It costs ten seconds, and it is a *diagnosis* rather than an attempt — its result names your next move instead of ending your turn. It is also legal for a specific reason: substituting works exactly when the function is continuous at a. That is precisely what rungs 1 and 2 have just ruled on, which is why rung 3 can be trusted the moment they are clear.

Why the algebraic $\frac{0}{0}$ moves are one move. Factoring, the conjugate, and combining a stacked fraction are doing the same job: exposing the factor $(x - a)$ that is sitting in both the top and the bottom. They differ only in how that factor is hiding. Not every $\frac{0}{0}$ is that shape — a trigonometric one is not, which is why it gets its own line on rung 4 and its own problem — but every *algebraic* one you will meet here is, and the limit is not finished until the *second* substitution returns a number.

Why rungs 2 and 6 need no insight. “Find k so the limit exists” is the equation *left limit* = *right limit*; you solve it like any other equation. Classifying a discontinuity is bookkeeping too — but it is bookkeeping in two stages, and running them together is the single most common way to name one wrong. The checklist establishes *that* the function broke. *How* it broke is a second question, and its answer comes from what the limit was doing, not from which line of the checklist you stopped on.

Where to Look When You’re Stuck

FIND THE SENTENCE THAT SOUNDS LIKE YOUR SITUATION

The tree above is for a problem you are about to start. This table is for one you are already inside. Find your sentence, do the thing in the middle column, turn to the page on the right. You do not have to have read this guide in order for it to work.

What is happening	First move	Where
“I’m not sure what it is even asking”	lim is the approach, $f(a)$ is the arrival. Two questions, one point.	p. 6, Prob. 10
“I substituted and got a number. Am I done?”	Only if one formula covers both sides of a. At a seam it answers with one piece and hides the other.	p. 8, Prob. 5
“I got $\frac{0}{0}$”	Not a verdict — a shape to match. Polynomial, root, stacked fraction, or trig?	p. 9, Prob. 2–4
“$\frac{0}{0}$, but with a sin or cos in it”	Stop factoring; nothing will come out. Match $\frac{\sin u}{u}$, or trap it and squeeze.	p. 10, Prob. 4
“I got $\frac{\text{nonzero}}{0}$”	Vertical asymptote. What’s left is the sign on each side, read off the surviving power.	p. 12, Prob. 7
“There is a seam, or a constant to find”	A one-sided limit from each piece’s *own* formula. Set them equal to pin the constant.	p. 11, Prob. 5
“x is going to $\pm\infty$”	Dominant term top and bottom — and settle $\sqrt{x^2} = \lvert x\rvert$ before anything else.	p. 13, Prob. 6
“The two sides disagree. What do I write?”	Both sides in one-sided notation, *then* the conclusion. “DNE” alone discards what you found.	p. 11, Prob. 10
“Is it continuous here?”	Three conditions in order: value, limit, agreement. All three or it isn’t.	p. 14, Prob. 5
“It is discontinuous. What do I call it?”	Not the checklist — what the *limit* did. Same finite number, different finite numbers, unbounded, or never settling.	p. 14, Prob. 7, 10
“I was handed a table”	Top half downward, bottom half *upward*. Both halves close in.	Prob. 9
“I was handed a graph”	The curve gives limits, the dots give values. An open circle means the curve arrived and the function didn’t.	Prob. 10
“It says *show that a value exists*”	IVT. Continuous and *why*, both endpoints with the target strictly between, then cite it.	p. 16, Prob. 8

If none of those is your sentence, the last section of this guide is the longer version. It is organized the same way — by what went wrong, not by what the topic is called.

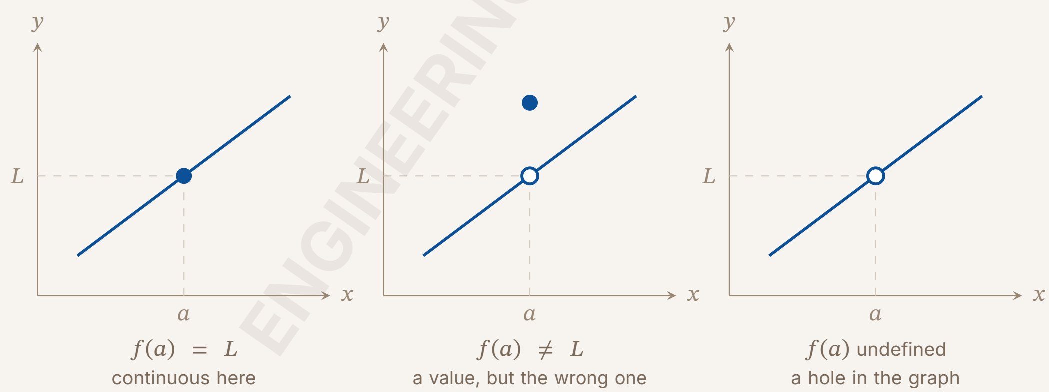
Master Toolbox — Everything These Problems Use

Read these once now, before you start the problems, so that you know what is in here. Then come back to them *while* you work — this section is reference, not narrative, and no box below depends on your having read the one above it.

THE ONE IDEA UNDER EVERYTHING

$\lim_{x\to a} f(x) = L$ means: as x gets arbitrarily close to a (from *both* sides), $f(x)$ gets arbitrarily close to L. The limit is about the **approach, never the arrival**.

Read that definition once more and notice what it never mentions: $f(a)$. The value the function takes *at* a is not part of the question being asked. It can equal L, it can be some other number entirely, or the function can fail to be defined there at all — and the limit is L in every one of those cases, because in every one of them the approach did exactly the same thing.



The approach is identical in all three — so $\lim_{x\to a} f(x) = L$ in all three. Only the arrival differs.

Those three pictures are the core of the unit in miniature. Every question about *continuity at a particular point* — does the limit exist there, is the function continuous there, what kind of discontinuity is this — is a question about how the second and third pictures differ from the first. Most limit mistakes are arrival answers to approach questions.

Not everything in the unit is that shape, and it is worth knowing which questions are not, so you don't go looking for a single bad point when there isn't one. A function can be broken at *two* points at once (Problem 7), and two of this unit's questions are not about a single point at all: end behavior asks what happens infinitely far out (Problem 6), and the Intermediate Value Theorem makes a claim about a whole interval (Problem 8). Rung 5 and rung 7 exist for exactly those.

THE LIMIT LAWS — AND THE ONE HYPOTHESIS THAT EVER FAILS

Limits get along with arithmetic. If $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, and *both of those are real numbers*, then

$$\lim_{x\to a}\big(f \pm g\big) = A \pm B, \qquad \lim_{x\to a}\big(c\,f\big) = c\,A, \qquad \lim_{x\to a}\big(f\,g\big) = A\,B,$$

$$\lim_{x\to a} \frac{f}{g} = \frac{A}{B} \quad \text{provided } B \neq 0, \qquad \lim_{x\to a}\big[f(x)\big]^n = A^n, \qquad \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{A}.$$

These are the reason substitution works at all. A polynomial is built out of constants and x by adding and multiplying, and the laws say a limit walks straight through both operations — so $\lim_{x\to a}$ of a polynomial is the polynomial evaluated at a, and every "just plug it in" you have ever done was these six lines quietly doing their job.

The quotient law is the one with a condition, and that condition is the entire unit. $B \neq 0$ is not fine print. When the bottom limit *is* zero, the law does not apply — and that is exactly the moment substitution hands you $\frac{\text{nonzero}}{0}$ or $\frac{0}{0}$ and the decision tree takes over. So the awkward cases in this guide are not places where the rules broke. They are places where a rule politely declined, and the rest of the packet is what you do instead.

Reading them backwards is a trap the exam likes. The laws run one direction only: if the pieces have limits, the combination does. *Not* the reverse. Both $\lim_{x\to 0} \frac{|x|}{x}$ and $\lim_{x\to 0}\left(-\frac{|x|}{x}\right)$ fail to exist, and their sum is 0 everywhere except the origin, so the sum's limit is a perfectly good 0. "$\lim(f+g)$ exists" tells you nothing about f and g on their own.

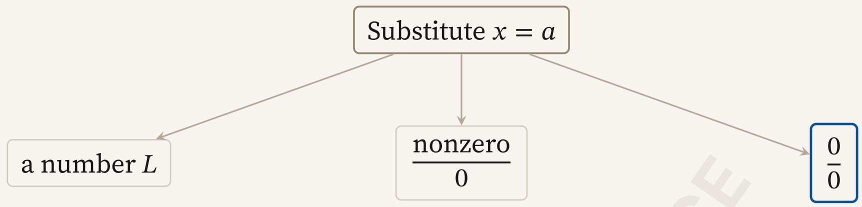
Where this shows up. Most often you are handed A and B — by a table, by a graph, or in a sentence — and asked to assemble something from them without ever seeing a formula. If $\lim_{x\to 2} f(x) = 3$ and $\lim_{x\to 2} g(x) = -5$, then

$$\lim_{x\to 2} \frac{f(x) + 2g(x)}{g(x)} = \frac{3 + 2(-5)}{-5} = \frac{-7}{-5} = \frac{7}{5},$$

and the only judgment involved was checking that the denominator's limit wasn't zero before using the quotient law. Say that check out loud; it is the part worth a point.

SUBSTITUTION IS A DIAGNOSIS, NOT AN ATTEMPT

Once rungs 1 and 2 are clear — you have a formula rather than a picture, and one formula rather than a piecewise rule with a seam at a — every limit in this guide starts the same way: put a in and see what comes back. That is not a guess, and it is not a first try that might fail. It is a *test*, and its result names your next move.



Stop. The limit is L. Nothing clever is left to do.

Asymptote. Algebra can't help. Analyze the sign on each side.

Indeterminate. Do algebra: factor, conjugate, or combine.

Why substitution works when it works. Plugging in is legal exactly when the function is *continuous* at a — and polynomials are continuous everywhere, while a rational function is continuous everywhere its denominator isn't zero. So for most of what you meet in this unit, "I can substitute" and "the function is continuous here" are the same sentence said two different ways. That is why the continuity material later in this guide is not a new topic bolted on at the end: it is the rule that made step one legal in the first place.

It is also the reason the seam question sits *above* this one on the ladder. A piecewise function has no single formula at its seam, so there is nothing there for "continuous at a" to be true of yet — and substituting anyway will hand you a number from whichever piece owns the endpoint, with no warning that the other piece went somewhere else entirely. Problem 5(c) is that exact number.

Outcome 1 — a number. You are done. Looking for something cleverer is how students turn a finished problem into a wrong one. If substitution returned $\frac{9}{6}$, the answer is $\frac{3}{2}$, and the fact that it felt too easy is not evidence of anything.

Outcome 2 — $\frac{\text{nonzero}}{0}$. The denominator is collapsing toward zero while the numerator is not. So you are dividing a number that stays put by a number that becomes arbitrarily small, and the quotient grows without bound. No algebra fixes this, because there is nothing to cancel — that zero is real and it isn't going anywhere. Your remaining job is a sign question: which direction does it blow up on each side?

Outcome 3 — $\frac{0}{0}$. Top and bottom are *both* collapsing, and the answer depends on which of them collapses faster — which is precisely the thing the symbol $\frac{0}{0}$ cannot tell you. This is called an **indeterminate form**, and the phrase means what it says:

not "undefined," but "not yet determined." Compare $\lim_{x\to 0} \frac{x}{x} = 1$, $\lim_{x\to 0} \frac{2x}{x} = 2$, and $\lim_{x\to 0} \frac{x^2}{x} = 0$ — all three substitute to $\frac{0}{0}$, and all three have different answers. The form is a question, not a result.

For the algebraic limits in this guide, both top and bottom hitting zero at $x = a$ means both of them contain the factor $(x - a)$. Finding it is the whole task.

THE $\frac{0}{0}$ PLAYBOOK

$\frac{0}{0}$ is an instruction to go looking. In the three *algebraic* shapes below the instruction is the same in substance — *expose the shared factor and remove it* — and only the technique changes, depending on how the factor is hiding. The fourth row is the exception worth knowing about before it turns up on a test: when the collapsing pieces are trigonometric, there is no shared factor to find, and no amount of algebra will produce one.

Shape you see	**Move**	**Why that move works**
polynomials	factor, cancel, substitute again	factoring *is* finding the shared factor
a square root	multiply by the conjugate	the difference of squares turns the root into a polynomial
fraction inside a fraction	combine into one fraction first	there is nothing to cancel until the top is one object
a sine or a cosine	a known special limit, or the squeeze theorem (Problem 4)	nothing cancels — $\sin x$ has no factor of x in it, so the answer has to come from a result you already have

Why canceling is legal. This is the step students feel guilty about, so it is worth being exact. When you write

$$\lim_{x\to a} \frac{(x-a)\,p(x)}{(x-a)\,q(x)} = \lim_{x\to a} \frac{p(x)}{q(x)}$$

you have not divided by zero. The limit only ever looks at x values *near* a and *never at* a *itself* — that is what "as x approaches a" means. For every one of those x, the quantity $x - a$ is a genuine nonzero number, and dividing by a nonzero number is something you have been allowed to do since seventh grade. The two functions above disagree at exactly one point, $x = a$, and a limit cannot tell them apart, because a limit never asks about that point.

The shape the problems in this guide don't show you. Complex fractions turn up on the exam and nowhere else in this packet, so here is the move in full. Suppose

$$\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$$

Substituting gives $\frac{0}{0}$, and there is nothing to factor and no root to conjugate — because the numerator isn't one object yet. Combine it over a common denominator first:

$$\frac{1}{x}-\frac{1}{2}=\frac{2-x}{2x}$$

so the whole expression becomes

$$\frac{2-x}{2x}\cdot\frac{1}{x-2}=\frac{-(x-2)}{2x\,(x-2)}=\frac{-1}{2x}$$

using $2-x=-(x-2)$ to make the shared factor visible. Now substitute again: $\frac{-1}{2(2)}=-\frac{1}{4}$. The pattern is the same one you have been running all along — the combining step existed only to produce something that *could* be canceled.

THE $\frac{0}{0}$ THAT ISN'T ALGEBRA: SQUEEZE AND THE TRIG LIMITS

Every other $\frac{0}{0}$ in this guide had a factor hiding in it. A trigonometric one does not. There is no x tucked inside $\sin x$ waiting to be pulled out, so factoring, conjugates, and combining all have nothing to grip — and a student who has only ever been told "$\frac{0}{0}$ means do algebra" will grind at $\frac{\sin x}{x}$ until the clock runs out. Two results replace the algebra. Both are worth memorizing outright:

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0$$

The first says something you can see: very near zero, $\sin x$ and x are the same number to as many decimals as you like. ($\sin(0.01)=0.00999983\ldots$) The graph of $\sin x$ leaves the origin along the line $y=x$ and only later peels away, so the ratio of the two starts at 1. That is also why the limit needs x in *radians* — in degrees the sine curve leaves the origin along a much flatter line and the ratio is $\frac{\pi}{180}$ instead.

Matching the pattern. The result is not really about x. It says $\frac{\sin u}{u}\to 1$ for *any* quantity u that is going to zero — the angle inside the sine and the thing underneath just have to be the same. So make them match, and pay for it with a constant:

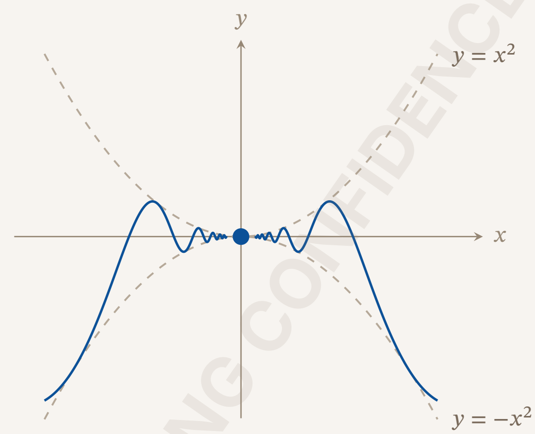
$$\lim_{x\to 0}\frac{\sin 5x}{3x}=\lim_{x\to 0}\frac{5}{3}\cdot\frac{\sin 5x}{5x}=\frac{5}{3}\cdot 1=\frac{5}{3}$$

Multiplying top and bottom by 5 costs nothing and buys the pattern. That one move handles nearly every trig limit the exam asks for.

The squeeze theorem. When a function is too wild to evaluate but you can trap it, trap it. If

$$g(x) \le f(x) \le h(x) \text{ for all } x \text{ near } a, \qquad \text{and} \qquad \lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L,$$

then $\lim_{x\to a} f(x) = L$ as well. The argument is not subtle: the two outer functions are closing in on the same place and f is pinned between them, so it has nowhere else to go.



$f(x) = x^2\cos\left(\frac{1}{x}\right)$ oscillates faster and faster as it nears the origin, so it has no pattern to read. It does not need one. It is caught between $-x^2$ and x^2, both of which go to 0, and the gap closes to nothing.

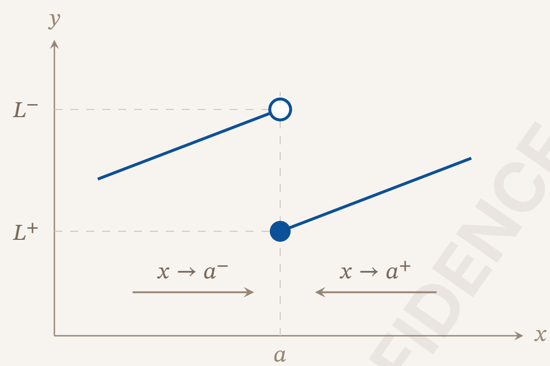
The trap to set is almost always the same one: a sine or a cosine is stuck between -1 and 1 no matter what you feed it, so multiply that sandwich by whatever is out front and you have your g and h. And note what the squeeze does *not* require — it never asks you to evaluate f, or even to know what f is doing. That is the whole point of using it.

ONE-SIDED LIMITS AND EXISTENCE

$\lim_{x\to a^-}$ means: approach a from the left, using only x values *less* than a. $\lim_{x\to a^+}$ uses only x values greater than a. The two-sided limit exists only when both of these arrive at the same place — and “a place” has to mean an actual number:

$$\lim_{x\to a} f(x) = L \quad \Longleftrightarrow \quad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L, \quad L \text{ a finite number}$$

That word *finite* is doing real work, and leaving it out is a mistake you can make while doing everything else right. Take $\frac{1}{x^2}$ at $x = 0$: both sides run to $+\infty$, so in a loose sense they "agree." But $+\infty$ is not a number the function is getting arbitrarily close to — it is shorthand for the function escaping every number you name. The two sides matching there does not rescue the limit; nothing does. Agreement is necessary, not sufficient.



Each side is a complete, well-behaved journey. They simply end in different places — so there is no single number the function is approaching, and $\lim_{x\to a} f(x)$ does not exist.

Notice what the picture does *not* show: anything wrong with either one-sided limit. Both exist, both are perfectly ordinary numbers. The failure is entirely in the comparison. That is why the honest way to report a nonexistent limit is to name both sides and then say they disagree — "does not exist" on its own throws away everything you actually found out.

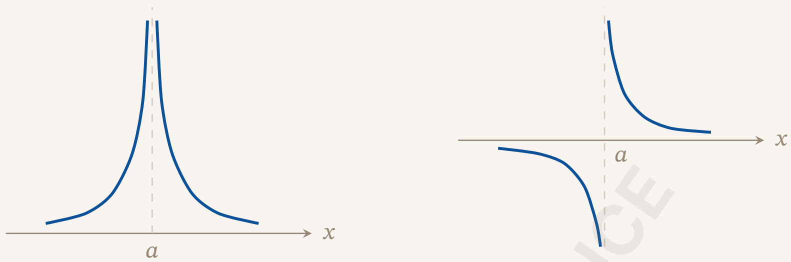
When a limit fails to exist, there are only three ways it can happen, and saying *which* is part of the answer:

- the two sides disagree — a **jump**;
- the function is unbounded — it runs off to $\pm\infty$;
- the function **oscillates** without settling, like $\sin(1/x)$ near 0, which crosses every value between -1 and 1 infinitely often no matter how close you get.

INFINITE LIMITS: WHEN THE DENOMINATOR ALONE GOES TO ZERO

This is outcome 2 from the diagnosis — $\frac{\text{nonzero}}{0}$ — and it means the graph has a **vertical asymptote** at $x = a$. Writing $\lim_{x\to a} f(x) = +\infty$ is not claiming the limit equals some number called infinity. It is a *description of how* the limit fails to exist: the function isn't wandering or jumping, it is climbing without bound. That description is worth more credit than "DNE," so give it.

How to find the sign on each side. You are not guessing. Look at the factor in the denominator that is producing the zero, and ask what sign it has just to the left of a and just to the right. If the fraction had a common factor and you canceled it first, read the power that is *left* in the denominator, not the one you started with — Problem 7 is exactly that case.



$\frac{1}{(x-a)^2}$ — **even** power
$(x - a)^2 > 0$ on both sides, so $+\infty$ both ways

$\frac{1}{x-a}$ — **odd** power
$(x - a)$ changes sign, so $-\infty$ left, $+\infty$ right

An even power of $(x - a)$ keeps its sign as you cross a; an odd power flips it. Count the power that survives after canceling, and the two sides are decided before you test a single number.

If counting powers feels abstract, there is a fallback that never fails: pick a number just to the left of a and one just to the right, and read the *signs* of the top and bottom separately instead of computing the value. Take $\frac{x-2}{x-4}$ near $x = 4$. At $x = 3.9$ the top is 1.9 — positive — and the bottom is -0.1 — negative and tiny — so the quotient is large and negative: the function runs to $-\infty$ from the left. At $x = 4.1$ the top is still positive and the bottom is $+0.1$, so the quotient is large and positive: $+\infty$ from the right. Write those two checks down. Doing them in your head is where this goes wrong.

LIMITS AT INFINITY: END BEHAVIOR AND THE $|x|$ TRAP

$x \to \pm\infty$ is a different question wearing the same symbol. Here nothing is dividing by zero and nothing is blowing up at a point — you are asking what the function *settles into* far out along the axis. A finite answer L means a **horizontal asymptote** $y = L$.

Dominant terms. Out at large $|x|$, the highest power in a polynomial is so much bigger than everything else that the rest are rounding error. In $5x^3 - 2x$ at $x = 1000$, the first term is 5×10^9 and the second is 2000: the second term is not small, it is *irrelevant*. So for a rational function, compare the top and bottom degrees:

- top degree < bottom degree ⇒ the limit is 0 (the denominator outruns the numerator);
- degrees equal ⇒ the ratio of the *leading coefficients* (they grow at the same rate, so the ratio settles);
- top degree > bottom degree ⇒ $\pm\infty$ (the numerator outruns the denominator — and now you must decide which sign).

Why that shortcut is legitimate. It is the result of dividing top and bottom by the highest power in the denominator. For example,

$$\frac{5x^3 - 2x}{7x^3 + 4} = \frac{5 - \frac{2}{x^2}}{7 + \frac{4}{x^3}} \longrightarrow \frac{5-0}{7+0} = \frac{5}{7}$$

because every term of the form $\frac{c}{x^n}$ goes to 0. The "circle the dominant term" rule is just this calculation with the zeroes already taken. If you ever doubt the shortcut on an unusual problem, do the division — it always works and it is never wrong.

The root trap. $\sqrt{x^2} = |x|$, not x. The square root symbol returns the nonnegative root, always, so for $x < 0$ you must write $\sqrt{x^2} = -x$ — which is a positive number, because x itself is negative. Concretely, $\sqrt{9x^2} = 3|x|$, and

$$3|x| = \begin{cases} 3x, & x > 0 \\ -3x, & x < 0 \end{cases}$$

This is the single most-missed step in the unit, and its consequence is that a function with a square root on top can have *two different* horizontal asymptotes — one as $x \to +\infty$ and a different one as $x \to -\infty$. Problem 6 draws that picture.

CONTINUITY AT A POINT: THE THREE-PART CHECKLIST

Informally, f is continuous at a if you can draw through the point without lifting your pencil. Formally — and this is the version that scores — all three of these must hold:

$$(1)\ f(a) \text{ is defined} \qquad (2)\ \lim_{x\to a} f(x) \text{ exists} \qquad (3)\ \lim_{x\to a} f(x) = f(a)$$

Run them in that order. Each one is checking something the previous one assumed: (1) asks whether there is an arrival at all, (2) asks whether there is an approach at all, and (3) asks whether the two agree. You need all three because any one of them can fail while the others hold.

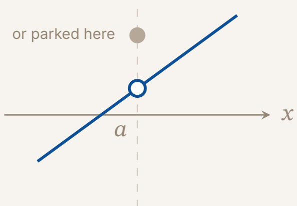
What the checklist does *not* do is name the break. It is tempting — three conditions, three kinds of discontinuity, surely they line up — and it is the most common way to classify one wrong. They do not line up. Look at the three pictures below and count: two of them fail condition (2), and the first one can fail either (1) or (3) depending on whether the value is missing or merely misplaced. Three failure modes, three names, no matching.

The reason is worth a sentence, because it makes the right method obvious. The checklist is a test for *continuity* — it is built to return yes or no, and it stops as soon as it has the answer. A classification is a different question asked afterward, and it has a different source of evidence: **the checklist settles whether the function broke; what the *limit* was doing settles how.**

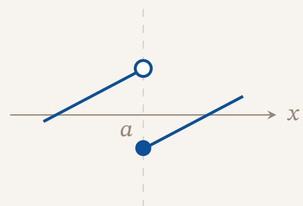
So run it in two stages. Stage one is the three conditions. If any fails, stage two is one more look — at the limit, not at the checklist:

What the limit did	Name	Repairable?
both sides reached the same finite number, but $f(a)$ is missing or sits elsewhere	removable (a hole)	yes — define $f(a)$ to be that number
the two sides reached *different* finite numbers	jump	no
the function ran off to $\pm\infty$ on at least one side	infinite (a vertical asymptote)	no
the function never settled — it kept oscillating	oscillating	no

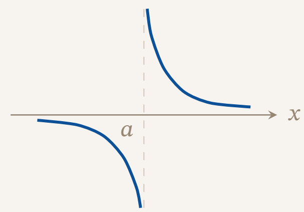
Two habits fall straight out of this. First, never name a discontinuity before you have computed the one-sided limits; the name *is* that computation. Second, be suspicious of any rule that lets you skip it. "$f(a)$ is undefined, so it's a hole" is the classic — and $\frac{1}{x}$ at $x = 0$ is the counterexample every AP class meets in the first week. Condition (1) fails there, exactly as it does at a hole. It is a vertical asymptote, because the limit is what tells you, and the limit ran away.



removable
the limit exists; the *value* is missing or sits somewhere else
(1) or (3) fails



jump
the one-sided limits disagree, so no limit exists at all
(2) fails



infinite
unbounded near a: a vertical asymptote
(2) fails

The name "removable" is literal, and it is worth knowing why. In the first picture you could *define* $f(a)$ to be the limit value, plug the hole, and the function would be continuous — one point repaired and the problem is gone. Nothing you can do to a single point repairs any of the others, which is what the last column of the table was recording. The fourth kind, oscillating, is the one you cannot draw here: near $x = 0$, $\sin(1/x)$ crosses every value between -1 and 1 infinitely often, so there is no picture with finitely many strokes in it.

INTERMEDIATE VALUE THEOREM

If f is **continuous on** $[a, b]$ and N is strictly between $f(a)$ and $f(b)$ —

$$\min\big(f(a),\ f(b)\big)\ <\ N\ <\ \max\big(f(a),\ f(b)\big)$$

— then $f(c) = N$ for some c in the *open* interval (a, b).

Those two strict inequalities are the price of the open interval, and they are cheap. If you let N equal an endpoint value, the guarantee still holds but only on the closed interval: $f(x) = x$ on $[0, 1]$ with $N = 0$ has exactly one solution, $c = 0$, and it sits at the edge rather than inside. Write the endpoint values with N caught strictly between them — as Problem 8 does, $-3 < 0 < 7$ — and the conclusion you want is the one you get.

Both hypotheses are load-bearing. *Continuity* is what forbids the function from skipping a value; without it the theorem says nothing at all, because a graph that is allowed to jump can step straight over N (Problem 8 draws exactly that). And N must actually be *caught between* the endpoint values — if it isn't, the theorem has no opinion.

What it never does. The IVT is a one-way guarantee. When its hypotheses hold, the value must exist; when they don't, it goes silent. Silence is not a "no." A student who concludes "the IVT doesn't apply, so there's no root" has read a theorem backwards, and that specific error is worth its own exam question — which is Problem 8(b).

What it never tells you either: *where* c is, or how many there are. It guarantees at least one. Finding it is a different problem with different tools.

The sentence that scores. A full-credit IVT justification names the continuity *with a reason*, shows the two endpoint values, and draws the conclusion:

h is continuous on $[1, 2]$ because it is a polynomial;
$h(1) = -3 < 0 < 7 = h(2)$;
so by the IVT there is a c in $(1, 2)$ with $h(c) = 0$.

Graders accept the argument without the theorem's name attached — but citing it costs you four words and removes all doubt, so cite it.

PROBLEM 1

Substitute first: three limits, three diagnoses

Evaluate each limit, or explain why it does not exist. *(a)* $\lim_{x\to 2} \dfrac{x^2+3x-1}{x+4}$ *(b)* $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$ *(c)* $\lim_{x\to 1} \dfrac{x+2}{(x-1)^2}$

BEFORE YOU COMPUTE

All three are rational functions and all three look alike on the page — there is no way to tell them apart by appearance, which is the entire point of putting them side by side. Substitute into each and *read the result* before doing anything else: a number, a $\frac{0}{0}$, and a $\frac{\text{nonzero}}{0}$ are three different instructions, and choosing your method before you have run the test is how students end up factoring something with no common factor.

WORKING

(a) Substitute and see what comes back:

$$\frac{(2)^2+3(2)-1}{2+4} = \frac{4+6-1}{6} = \frac{9}{6} = \frac{3}{2}$$

A number, so we are done — and it is worth saying why we are allowed to stop. The denominator $x + 4$ equals 6 at $x = 2$, not zero, so this rational function is continuous at

$x = 2$, and for a continuous function the limit *is* the value. There is no hidden second step. Resist the feeling that a calculus problem owes you more work than this.

(b) Substitute: $\frac{9-9}{3-3} = \frac{0}{0}$. Both top and bottom vanish at $x = 3$, which is the form's way of telling you they share the factor $(x-3)$. The numerator is a difference of squares, so finding it takes one line:

$$x^2 - 9 = (x-3)(x+3)$$

$$\lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3} (x+3)$$

The cancellation is legal because the limit only looks at x near 3, never at 3 itself — so $x - 3$ is a genuine nonzero number every time we divide by it. Now substitute the *second* time, which is the step people forget:

$$\lim_{x\to 3} (x+3) = 3 + 3 = 6$$

(c) Substitute: $\frac{1+2}{(1-1)^2} = \frac{3}{0}$. This is *nonzero* over zero — a different animal from (b). Nothing cancels, because the numerator does not vanish at $x = 1$; the function is genuinely unbounded near $x = 1$, and there is a vertical asymptote there.

So the remaining question is which direction, on each side. The numerator $x + 2$ is near 3 — positive — on both sides. The denominator $(x-1)^2$ is a square, so it is positive on *both* sides of 1 and shrinking toward zero. Positive over small-and-positive is large and positive, from either direction:

$$\lim_{x\to 1^-} \frac{x+2}{(x-1)^2} = +\infty \qquad \lim_{x\to 1^+} \frac{x+2}{(x-1)^2} = +\infty$$

Both sides agree on the behavior, so we can write a single statement:

$$\lim_{x\to 1} \frac{x+2}{(x-1)^2} = +\infty$$

Say clearly what that means: the limit does not exist as a number, and $+\infty$ is the honest description of *how* it fails.

ANSWER

(a) $\frac{3}{2}$ (b) 6 (c) $+\infty$ (unbounded — the limit does not exist as a number)

WATCH OUT

$\frac{0}{0}$ and $\frac{\text{nonzero}}{0}$ look like cousins and behave like strangers. $\frac{0}{0}$ means *do algebra — there's a hidden common factor*. $\frac{\text{nonzero}}{0}$ means *no algebra will save this — it's an asymptote; go*

> *analyze signs*. Confusing the two sends you factoring something that has no common factor, or writing ∞ where the answer is 6.
>
> The other trap in (c) is the exponent. Had the denominator been $(x-1)$ instead of $(x-1)^2$, the two sides would *not* agree: the factor would be negative on the left and positive on the right, giving $-\infty$ and $+\infty$, and writing a single two-sided answer would be wrong. The square is doing real work here. Never carry a one-sided sign conclusion across an example without re-checking the power.

ABOUT THIS EXCERPT

This is the opening of a 42-page guide: the diagnostic tree, the full Master Toolbox, and the first worked problem. 9 more problems follow in the complete guide, each worked the same way — what to notice before you start, every step shown, and the mistake that problem invites. The complete guide is shared with families during the fit conversation.

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