Engineering Confidence

Differentiation: Definition & Fundamental Properties

A worked guide to choosing the next move

AP CALCULUS AB · UNIT 2

Rates of change · The derivative by definition · Tangent lines · Differentiability · The rules: power, sum, product, quotient · sin, cos, e^x, $\ln x$, and the other four trig functions

A reference for the whole unit — not a single session's homework, but built the way my session guides are built: tool selection first, every step shown, commentary where the thinking usually goes wrong.

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READ THIS FIRST

About this guide. One sentence runs under everything in Unit 2: *a derivative is the limit of an average rate of change, and every rule in this unit is that limit computed once and remembered*. Unit 1 taught you to work at $\frac{0}{0}$; the derivative is the $\frac{0}{0}$ the whole subject is built on — the difference quotient $\frac{f(x+h)-f(x)}{h}$ substitutes to it every time — and the rules exist so that you never have to resolve that limit twice for the same kind of function.

The College Board's own note on this unit is about *precision*, not ideas: students "drop important notation, such as a parenthesis, or misapply the product rule by taking the derivative of each factor separately and then multiplying those together." Unit 2 is where the habit of self-correcting is installed, and this guide is built so that the check is part of the method rather than an afterthought — every worked problem ends by confirming its answer a second way.

Three mistakes to learn to catch:

- **Computing the right number without the set-up that scores.** A derivative estimated from a table earns its point for the written difference quotient, not for the number; a product rule earns its point for the structure written before the values go in. The exam says so in as many words.
- **Treating rules as independent facts.** The product rule is not a formula to memorize beside the power rule; it is the limit definition applied to a product, and a student who knows why it has two terms never multiplies the derivatives.
- **Forgetting what a derivative *is* while computing one.** $f'(3) = 8$ is a slope, a rate, and the limit of average rates over intervals closing on 3 — three sentences the exam asks for in words, with units.

Every topic in the unit, and where it lives. Ten of them:

- **2.1** average and instantaneous rates of change — the first card (p. 7) and Problem 1.
- **2.2** the derivative as a function, its notation, and the tangent line — two cards (p. 8); Problems 2 and 3.
- **2.3** estimating a derivative from a table or a graph (p. 10); Problem 4.
- **2.4** differentiability and continuity — when a derivative does not exist (p. 11); Problem 5.
- **2.5, 2.6** the power rule and the constant, sum, difference and constant-multiple rules (p. 12); Problem 6.
- **2.7** $\sin x$, $\cos x$, e^x, $\ln x$ (p. 14), and a limit that is a derivative in disguise (p. 15); Problems 7 and 8.
- **2.8** the product rule (p. 16); Problem 9.
- **2.9** the quotient rule (p. 17); Problem 10.
- **2.10** tangent, cotangent, secant, cosecant (p. 18); Problem 11.

The pages that follow are your **concept reference**: a diagnostic decision tree and a Master Toolbox, both built to be opened at the card you need rather than read front to back. What matters more is the order you work in. Try each problem *before* you read its solution — the "Before you compute" notes are placed to catch you at the exact moment a wrong turn usually happens. What eleven problems *cannot* do is give you enough repetitions; each is the clearest instance of its shape, so a problem you found easy is a signal to go find five more like it. Fluency is volume, and volume is homework.

WHERE THE POINTS GO ON THIS UNIT

The structure is the point. The College Board's own example: to estimate $C'(3.5)$ from a table, "students must present a difference quotient: $C'(3.5) \approx \frac{C(4)-C(3)}{4-3}$" — and "failure to present this structure will cost students the point they might have earned, even with a correct numerical answer." The same for a product: $f'(3) = u(3)\,v'(3) + v(3)\,u'(3)$ written out, *then* the values imported. Write the shape, then the numbers, every time. Problems 4 and 9 are built around exactly those two sentences.

Precision is scored. A dropped parenthesis turns $2(x+1)$ into $2x+1$; a product rule with one term is a wrong derivative, however clean the algebra after it. The exam's word is "self-correct": check the answer a second way before moving on, and this guide shows a second way on every problem.

The calculator's answer is not the answer. Where a calculator is allowed, the expression evaluated is written before the number, and the number is rounded (or truncated) to three places after the decimal point, as the question specifies. Store intermediate values; round once, at the end.

A derivative is a rate with units. "$h'(2) = -32$" is a number; "at $t = 2$ seconds the height is decreasing at 32 feet per second" is the answer to the question that was asked. The verbal representation is one of the four the CED names, and it is the one students skip.

Differentiability is justified, not asserted. "Not differentiable because there is a corner" is a picture; "f is continuous at 2, and $\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} = -1 \neq 1 = \lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h}$, so $f'(2)$ does not exist" is a justification. Unit 1's rule stands: hypotheses first, then the conclusion. And the margin that buys a 5 is the same as it was in Unit 1 — with the 2025 average free-response score, eight missed multiple-choice questions; with a perfect one, thirty-four.

Diagnostic Decision Tree

HOW TO READ THE PROMPT

1. **A rate over an interval, or at an instant?** → Interval: the average rate, a difference quotient with the two endpoints. Instant: the derivative — the limit of that quotient. Say which one the words asked for before you compute.

2. **Handed a table or a graph, not a formula?** → Estimate. Table: the difference quotient across the nearest points, *written out*. Graph: draw the tangent and read its rise over run. Say "approximately."

above: what am I holding? — below: which tool does it want?

3. **"Use the definition"?** → The limit of the difference quotient, either form. It substitutes to $\frac{0}{0}$ by construction; finish it with Unit 1's algebra. No rule may appear.

4. **Does the derivative exist here?** → Continuity first — no continuity, no derivative. Then the one-sided difference quotients: unequal is a corner, infinite is a cusp or a vertical tangent.

5. **Which rule? Name the *outermost* operation.** →
   A sum or difference — term by term.
   A constant times something — keep the constant, differentiate the rest.
   x^r (rewrite roots and reciprocals first) — the power rule.
   sin, cos, e^x, $\ln x$ — the four memorized.
   Two things multiplied — the product rule.
   One thing over another — the quotient rule.
   tan, cot, sec, csc — sine and cosine, then quotient.

6. **A tangent line (or a normal)?** → Two numbers: the point from f, the slope from f'. Point–slope form, and leave it there.

7. **A limit that looks like a difference quotient?** → It is a derivative in disguise. Name f and a, then answer $f'(a)$ from a rule.

Rungs 1 and 2 ask what kind of object you are holding; from rung 3 down, stop at the first "yes." Rung 5 is where most of the unit's arithmetic lives, and its first line is the one to say out loud: *the outermost operation decides the rule*.

Three of those rungs hide a reason worth carrying.

Why the outermost operation decides. $x^2 \sin x$ is a *product*, so the product rule applies, and *inside* it the power rule and the sine rule handle the factors. $\frac{x^2}{\sin x}$ is a quotient of the same two pieces and takes the quotient rule. Read the expression from the outside in, name the last operation that was performed to build it, and that operation names the rule. Students who read from the inside out apply the power rule to x^2, the sine rule to $\sin x$, and then have two derivatives and no idea how to combine them — which is precisely the error the College Board describes.

Why continuity comes before any slope. A derivative at a is a limit of average rates *from* $f(a)$; if $f(a)$ is missing or the function jumps there, the average rates have nothing to converge to, and the question of a slope is over before it starts. So rung 4 begins with Unit 1's checklist and only then computes the one-sided quotients. The reverse order — slopes first, continuity as an afterthought — is how students report a derivative at a point where the function does not exist.

Why "use the definition" forbids the rules. The point of the question is the limit, and the rules are what the limit produces; using a rule to evaluate a definition problem is answering with the conclusion and skipping the argument, and it is scored as skipping the argument. The algebra is Unit 1's — expand, cancel the h, substitute — and it is never as long as it looks.

Where to Look When You're Stuck

FIND THE SENTENCE THAT SOUNDS LIKE YOUR SITUATION

The tree above is for a problem you are about to start. This table is for one you are already inside. Find your sentence, do the thing in the middle column, turn to the page on the right.

What is happening	First move	Where
"Average or instantaneous — I can't tell which it wants"	An interval in the words means average, a difference quotient with both endpoints. A single moment means the derivative.	p. 7, Prob. 1
"I plugged in $h = 0$ and got $\frac{0}{0}$"	Correct — that is what the definition does. Expand, cancel the h, substitute again.	p. 8, Prob. 2
"I have a table and it wants $f'(3.5)$"	The nearest two points around 3.5, the quotient written out with the function's name, then $\approx$.	p. 10, Prob. 4
"Is it differentiable there?"	Continuity first. Then the slope from each side; equal and finite, or it isn't.	p. 11, Prob. 5
"There's a square root or a $1/x^n$"	Rewrite as a power of x before touching a rule.	p. 12, Prob. 6
"I don't know what to do with sin, e^x, ln"	Four derivatives, memorized; the sign on cos is the one that leaves.	p. 14, Prob. 7
"This limit has $h \to 0$ but no f was given"	It is a derivative at a point. Find the f and the a hiding in it.	p. 15, Prob. 8
"Two things multiplied"	Product rule, two terms, structure written before values.	p. 16, Prob. 9
"One thing over another"	Quotient rule — and the minus sign has a side. Or rewrite with a negative power when the bottom is a monomial.	p. 17, Prob. 10
"tan, sec, cot, csc"	Sine and cosine underneath, quotient rule on top — or the four results, memorized with their signs.	p. 18, Prob. 11
"It wants the tangent line"	$y - f(a) = f'(a)(x - a)$. The point from f, the slope from f'.	p. 10, Prob. 3
"My answer is a number and the question was in words"	Say the rate, with its units, in a sentence. That is the answer.	p. 7, Prob. 1

If none of those is your sentence, the last section of this guide is the longer version, organized by what went wrong.

Master Toolbox — Everything These Problems Use

Use these cards as you work. Name the decision you need to make, then open the relevant card. Each card stands on its own; you do not need to read the whole toolbox before attempting a problem.

AVERAGE RATE, INSTANTANEOUS RATE, AND THE TWO DIFFERENCE QUOTIENTS

An **average rate of change** over an interval is a slope between two points on the graph — a difference quotient. The CED writes it two ways, and both appear on the exam:

$$\frac{f(a+h)-f(a)}{h} \qquad \text{and} \qquad \frac{f(x)-f(a)}{x-a}.$$

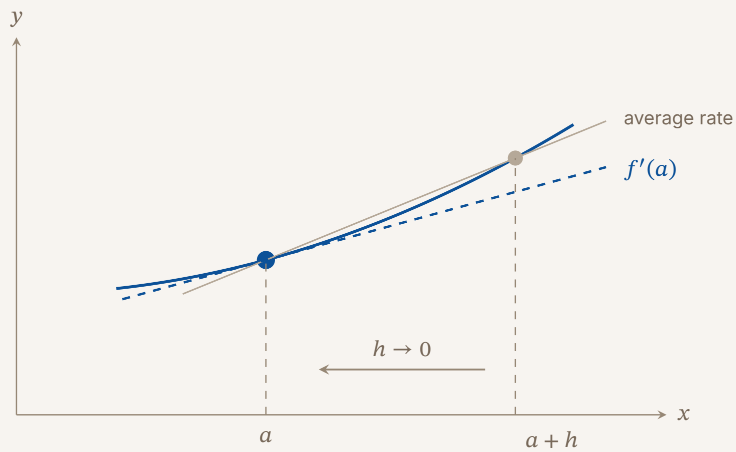
They are the same fraction. In the first the interval runs from a to $a+h$ and its width is h; in the second it runs from a to x and its width is $x-a$. Choose whichever makes the algebra cleaner and say which you chose.

The **instantaneous rate of change** at a is what those averages approach as the interval shrinks onto a — Unit 1's opening question, now given a name and a symbol:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a},$$

provided the limit exists. That proviso is topic 2.4, and it is not fine print.

Three sentences, one number. $f'(a)$ is the slope of the tangent line at $x=a$; it is the instantaneous rate of change of f at a; it is the limit of the average rates over intervals containing a. The exam asks for all three in words, and it asks for *units*: if $s(t)$ is feet and t is seconds, then $s'(2)$ is in feet per second, and the sentence "at $t=2$ the position is changing at 32 ft/s" is the answer, not 32.



The secant's slope is the average rate over $[a, a+h]$. Slide the second point toward the first and the secant turns into the tangent — its slope is $f'(a)$, the instantaneous rate.

A check first. One question opens each of the next three cards and is answered where that card ends. Answer it before you read on; getting it wrong is the point, because that is what makes the card stick.

1. TEST THE GRAPH CLAIM

If f has a horizontal tangent at $x = 0$, must the graph of f' cross the horizontal axis there, or only touch it? Decide first, then test your answer on $f(x) = x^3$.

THE DERIVATIVE AS A FUNCTION, AND THE FOUR WAYS TO WRITE IT

Do the limit at a general x instead of a particular a and the result is a new function, f':

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \qquad \text{provided the limit exists.}$$

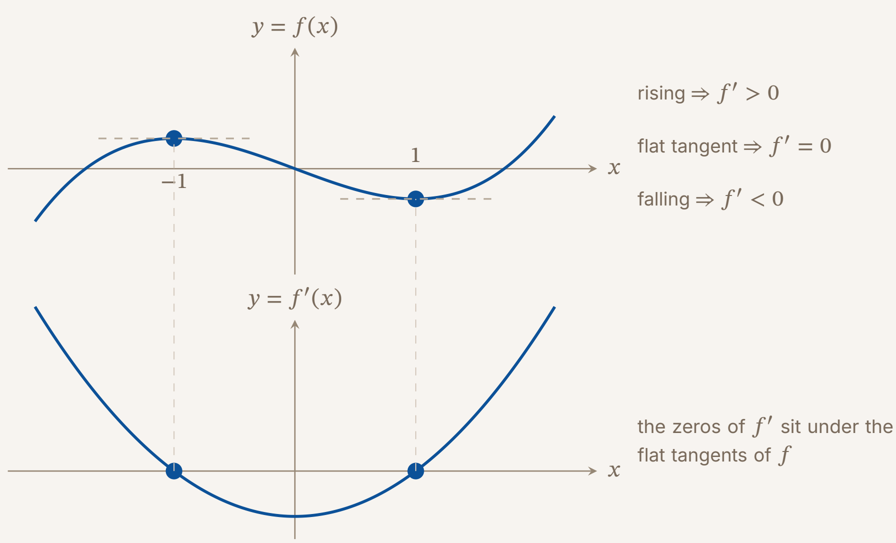
Its value at any x is the slope there. Here it is for $f(x) = x^2$, worked the only way Unit 2 allows until the rules are earned:

$$\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{h\,(2x+h)}{h} = 2x + h \quad \longrightarrow \quad f'(x) = 2x.$$

Substituting $h = 0$ at the start gives $\frac{0}{0}$; canceling the h first gives $2x + h$, which substitution can read. That is Unit 1's factoring move, and every "use the definition" problem is exactly it.

Four notations, one object. For $y = f(x)$ the derivative is written $f'(x)$, y', $\dfrac{dy}{dx}$, or $\dfrac{d}{dx}[f(x)]$. The last two are Leibniz's, and they are worth understanding rather than tolerating: $\frac{dy}{dx}$ is $\frac{\Delta y}{\Delta x}$ with the limit already taken — a slope written as a fraction whose parts have gone to zero together. $\frac{d}{dx}$ on its own is an instruction ("take the derivative with respect to x") waiting for something to act on. The exam uses all four without warning; the CED's skill 4.C is literally "use appropriate mathematical symbols and notation."

Four representations. A derivative can be given as a formula, as a slope on a graph, as a number estimated from a table, or as a sentence about a rate. The picture below is the one students misread most: the graph of f' is not a picture of f — it is a picture of f's *slopes*. Where f rises, f' is positive; where f has a horizontal tangent, f' is zero (it need not cross the axis); where f falls, f' is negative.



$f(x) = \frac{1}{3}x^3 - x$ above, $f'(x) = x^2 - 1$ below. Read the bottom graph as a report on the top one's slopes. The most common misreading is to look at the bottom and describe it as though it were f.

CHECK 1 — TEST THE GRAPH CLAIM

Only touch. A horizontal tangent at 0 says $f'(0) = 0$, and that is the whole of what it says: the graph of f' has to *reach* the axis there, not go through it. Take $f(x) = x^3$. Its tangent at the origin is flat, and $f'(x) = 3x^2$ is zero at 0 and positive on both sides — the parabola touches the axis and turns back up. Crossing would mean f' changes sign, which is a claim about f turning around, and x^3 never turns around. The figure above happens to show a case where f' does cross, because $\frac{1}{3}x^3 - x$ really does have a maximum and a minimum; do not read the picture as the rule.

THE TANGENT LINE — TWO NUMBERS AND A FORM

The tangent line at $x = a$ passes through the point $(a, f(a))$ with slope $f'(a)$. Point–slope form was built for exactly this:

$$y - f(a) = f'(a)(x - a).$$

Two numbers are all it needs, and they come from two different places: the point from f, the slope from f'. Mixing them up — using $f(a)$ as the slope, or $f'(a)$ as the height — is the whole error class on tangent-line questions. Leave the answer in point–slope form unless the question asks for another; rearranging is a place to drop a sign for no credit.

The **normal line** is perpendicular to the tangent at the same point: same point, slope $-\frac{1}{f'(a)}$ (and vertical if $f'(a) = 0$).

Why the tangent matters beyond the question that asks for it. Zoom in on a differentiable point and the curve straightens — close enough, the function and its tangent line are the same line. That is what differentiability *means* geometrically (*local linearity*), and it is why Unit 4 will use the tangent line to approximate f near a: $f(x) \approx f(a) + f'(a)(x - a)$. The tangent line is the derivative made visible.

ESTIMATING A DERIVATIVE FROM A TABLE, A GRAPH, OR A CALCULATOR

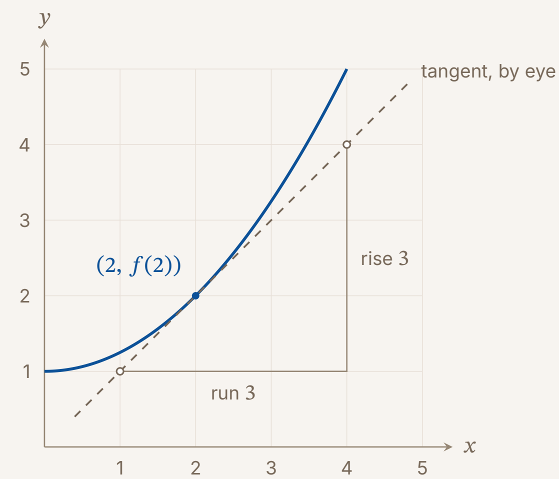
When there is no formula there is no rule to apply, and the derivative is *estimated* from the average rate over the smallest interval the data allows.

From a table. For $f'(c)$, take the two tabulated points that bracket c most tightly and write the difference quotient *with the function's name in it*:

$$f'(3.5) \approx \frac{f(4) - f(3)}{4 - 3}.$$

That line is the point. The College Board says so directly: a correct number without this structure loses the credit. If c is itself a tabulated point, use the points on either side of it (the symmetric difference $\frac{f(c+k)-f(c-k)}{2k}$) — it is a better estimate, and it is the one graders expect when both neighbors are available. Write $\approx$, never $=$: a table is evidence, not a formula.

From a graph. Draw the tangent by eye at the point, then find two places where your line crosses grid intersections and read rise over run between them. Below, the tangent at $(2, f(2))$ passes through $(1, 1)$ and $(4, 4)$: rise 3 over run 3, so $f'(2) \approx 1$. Say "approximately," and expect a tolerance — the grader has one, and a line drawn by eye that reads 0.9 or 1.1 is a correct answer. What is not correct is the slope of a chord that starts at the point and ends somewhere on the curve; that is an average rate, and the question asked for the tangent's.

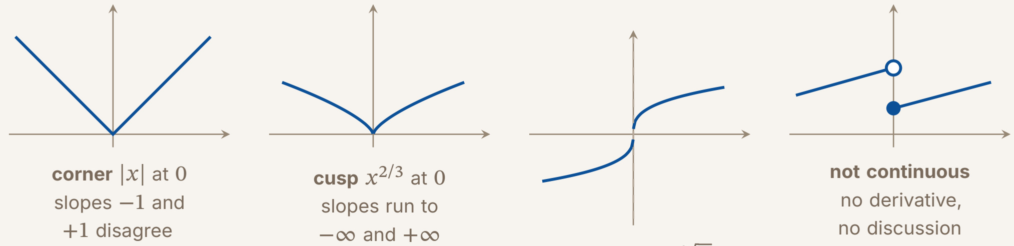


The curve is f; the dashed line is the tangent drawn at $x = 2$. The two open circles are where it crosses grid intersections, and $\frac{4-1}{4-1} = 1$ is the estimate. Use the line's crossings, never two points on the curve.

With technology. A calculator's numerical derivative (nDeriv, or $\frac{d}{dx}$ at a value) is allowed on the calculator sections; the expression you evaluated is written on the page, and the result is rounded to three places after the decimal point unless told otherwise. Store intermediate values rather than retyping rounded ones.

DIFFERENTIABILITY AND CONTINUITY: WHEN A DERIVATIVE DOES NOT EXIST

Differentiable implies continuous. If $f'(a)$ exists, then f is continuous at a — so a point not in the domain of f is not in the domain of f', and a jump, a hole, or an asymptote at a ends the question. The CED states this one direction only, and the direction matters: **continuous does not imply differentiable.** A function can pass Unit 1's checklist at a and still have no slope there, in three ways.



corner $|x|$ at 0
slopes -1 and $+1$ disagree

cusp $x^{2/3}$ at 0
slopes run to $-\infty$ and $+\infty$

vertical tangent $\sqrt[3]{x}$ at 0
one slope, and it is infinite

not continuous
no derivative, no discussion

The test that scores. Differentiability at a is a limit, so it is settled by one-sided limits of the difference quotient, exactly as a limit's existence was settled in Unit 1:

$$\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

must both exist and agree. For $|x|$ at 0 the left quotient is $\frac{-h}{h} = -1$ and the right is $\frac{h}{h} = 1$: a corner. For $\sqrt[3]{x}$ at 0 the quotient is $\frac{h^{1/3}}{h} = h^{-2/3}$, which runs to $+\infty$ from both sides: a vertical tangent, the CED's own example. A picture of a corner is not a justification; those two limits are.

Piecewise functions. To be differentiable at a seam a function must first be continuous there (Unit 1's seam equation) and then have matching one-sided slopes — two equations, and "find m and b so that f is differentiable at 1" is asking for both. Solving only the slope equation, or only the continuity one, is half credit on a good day.

2. PRESERVE THE DOMAIN

Simplify and differentiate $f(x) = x^3/x$. Does the simplified expression define the original function at zero?

THE POWER RULE — AND THE REWRITING THAT HAS TO HAPPEN FIRST

For any real exponent r,

$$\frac{d}{dx}[x^r] = r\,x^{r-1}.$$

Bring the exponent down as a coefficient; subtract one from the exponent. It is the definition, computed once: for x^2 you saw $2x$ above; for x^3 the same expansion gives $\frac{3x^2h+3xh^2+h^3}{h} = 3x^2 + 3xh + h^2 \to 3x^2$; and the pattern the binomial expansion produces is rx^{r-1} every time, which is why one rule covers every power.

Rewrite before you differentiate. The rule reads x^r and nothing else, so every root and every reciprocal is rewritten as a power *first*:

$$\sqrt{x} = x^{1/2}, \qquad \frac{1}{x^3} = x^{-3}, \qquad \frac{1}{\sqrt{x}} = x^{-1/2}, \qquad \sqrt[3]{x^2} = x^{2/3}.$$

Then the rule, then (if the question wants it) rewrite back: $\frac{d}{dx}\sqrt{x} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$. Students who try to differentiate $\sqrt{x}$ "as a root" invent a rule; students who rewrite never need one.

The sign on a negative exponent. $\frac{d}{dx}\left[x^{-3}\right] = -3x^{-4}$: the exponent goes *more* negative, from -3 to -4, because subtracting one from a negative number makes it smaller. Writing $-3x^{-2}$ is the standard slip, and it comes from "subtract one" being heard as "move toward zero."

Constants. A constant has no slope: $\frac{d}{dx}[7] = 0$. A constant *times* x has slope equal to the constant: $\frac{d}{dx}[7x] = 7$. And π^2, e^3, $\sqrt{5}$ are constants too, however much they look like functions — their derivatives are zero.

CHECK 2 — PRESERVE THE DOMAIN

No. For every $x \neq 0$, $x^3/x = x^2$, so $f'(x) = 2x$ there — and $x = 0$ was never in the original function's domain, so it is not in the derivative's either. Rewriting is a move you make on the expression; it does not hand back an input the function could not accept. The graph is the parabola $y = x^2$ with a hole at the origin, and the hole survives every simplification you do to it. Write the restriction down when you rewrite — Problem 6(c) is this same shape with the point of the question hidden in it.

CONSTANT MULTIPLES, SUMS, AND DIFFERENCES — AND THE TWO RULES THAT DO NOT EXIST

Derivatives pass through addition, subtraction, and constant multiples:

$$\frac{d}{dx}\left[c\,f(x)\right] = c\,f'(x), \qquad \frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x).$$

These are Unit 1's limit laws wearing new clothes: a limit walks through a sum and through a constant factor, and the derivative is a limit. So a polynomial is differentiated term by term, each term by the power rule, each coefficient carried along:

$$\frac{d}{dx}\left[4x^3 - 2x^2 + 7x - 9\right] = 12x^2 - 4x + 7.$$

The two rules that do not exist. The derivative of a product is *not* the product of the derivatives, and the derivative of a composition is *not* the derivative of the outside evaluated at the inside. Test the first on $x \cdot x$: the true derivative of x^2 is $2x$, and the product of the derivatives is $1 \cdot 1 = 1$. Wrong on the simplest case there is. The product rule (two cards down) is what the limit actually produces; the chain rule for compositions is Unit 3's, and until then $(2x+1)^5$ is handled by expanding it or by waiting — never by "$5(2x+1)^4$" alone, which is the chain rule with its second factor missing and is marked as such.

THE FOUR MEMORIZED DERIVATIVES: $\sin x$, $\cos x$, e^x, $\ln x$

$$\frac{d}{dx}[\sin x] = \cos x, \quad \frac{d}{dx}[\cos x] = -\sin x, \quad \frac{d}{dx}[e^x] = e^x, \quad \frac{d}{dx}[\ln x] = \frac{1}{x}\ (x > 0).$$

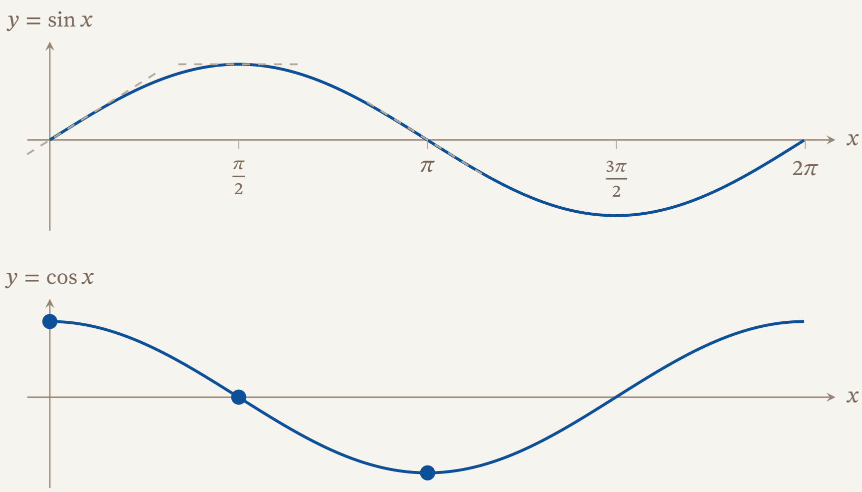
Four facts, and the sign on the second is the one that leaves: cosine's derivative is *negative* sine. Say it aloud when you write it.

Where the sine rule comes from — because a rule you can rebuild is a rule you cannot lose. Apply the definition and the angle-addition identity $\sin(x+h) = \sin x \cos h + \cos x \sin h$:

$$\frac{\sin(x+h) - \sin x}{h} = \sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h}.$$

As $h \to 0$ the first fraction goes to 0 and the second to 1 — Unit 1's two memorized trig limits, which were never really about limits. They were the derivative of sine at zero, waiting. So the whole thing goes to $\sin x \cdot 0 + \cos x \cdot 1 = \cos x$.

Where the exponential rule comes from. $\frac{e^{x+h} - e^x}{h} = e^x \cdot \frac{e^h - 1}{h}$, and the second factor goes to 1 — check it: $h = 0.01$ gives $\frac{1.01005 - 1}{0.01} = 1.005$. That limit being exactly 1 is what makes e the base calculus prefers: the function whose rate of change is itself. The logarithm's rule waits for Unit 3, where it falls out of e^x by the inverse relationship; until then it is a fact to carry, with its domain — $\ln x$ lives on $x > 0$, so $\frac{1}{x}$ as its derivative means the positive branch only.



Three tangents drawn on $\sin x$: slope 1 at 0, slope 0 at $\frac{\pi}{2}$, slope -1 at π. Read those three slopes off the bottom graph — they are $\cos 0$, $\cos\frac{\pi}{2}$, $\cos\pi$. The derivative of sine is not a coincidence you memorize; it is the slopes of sine, plotted.

A LIMIT THAT IS A DERIVATIVE IN DISGUISE

The CED's topic 2.7 carries a second learning objective the title hides (LIM-3.A): *recognize* the definition of the derivative when a limit is written in its shape, and evaluate the limit by a rule instead of by algebra. The tell is the form:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \qquad \text{or} \qquad \lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

Find the f and the a. In $\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{6}+h\right)-\frac{1}{2}}{h}$ the function is sin, the point is $\frac{\pi}{6}$ (and $\frac{1}{2}$ is $\sin\frac{\pi}{6}$, which is the confirmation), so the limit is $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. No algebra, no identity — a rule. In $\lim_{x\to 2}\frac{x^5-32}{x-2}$ the function is x^5 at $a = 2$, and the answer is $5\cdot 2^4 = 80$; try that one by factoring and you will see why recognizing is worth learning.

The trap is a limit that *almost* has the shape: $\frac{f(a+h)-f(a-h)}{h}$ is $2f'(a)$, not $f'(a)$; $\frac{f(a+h)-f(a)}{2h}$ is $\frac{1}{2}f'(a)$. Match the pieces exactly before you name the derivative.

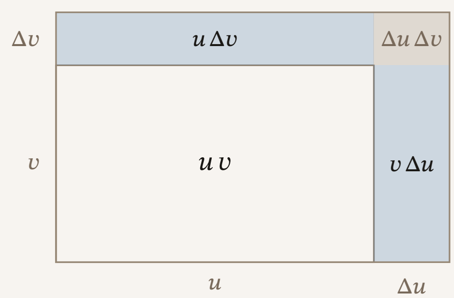
3. EXPLAIN THE MISSING TERM

In the product rectangle the small corner is $\Delta u\,\Delta v$. After the whole increment is divided by h, why does the corner's contribution vanish for differentiable factors — and does that mean $u'v'$ is zero?

THE PRODUCT RULE — WHY IT HAS TWO TERMS

$$\frac{d}{dx}[u\,v] = u'\,v + u\,v'.$$

Each factor gets its turn to change while the other holds still, and the two contributions add. The picture makes it obvious. Let u and v be the sides of a rectangle, so uv is its area; grow x a little and each side grows a little:



The new area is $uv+v\,\Delta u+u\,\Delta v+\Delta u\,\Delta v$. Divide the change by $h = \Delta x$ and let $h \to 0$: the strips contribute $vu' + uv'$, while the corner contributes $(\Delta u/h)\Delta v \to u' \cdot 0 = 0$.

A product changes for two reasons — u moves, and v moves — so its derivative has two terms, one for each. The corner is the piece that is neither, and it is worth being exact about how it leaves. Divide the area change by h and the corner becomes $\Delta u\,\Delta v/h = (\Delta u/h)\,\Delta v$; for differentiable factors the first part goes to u' while the second goes to 0, so the product goes to 0. Note what that argument does *not* say. It does not say $u'v'$ vanishes — $u'v'$ is generally a perfectly good nonzero number, and on $u = v = x$ it is 1. It says only that $u'v'$ is not the rate of change of uv, which is the whole reason the wrong rule is wrong. The College Board names this error in the unit's own overview; it is the most common wrong derivative on the exam that is not an arithmetic slip.

The structure before the values. On the exam, a product rule at a point is written as a structure first — $f'(3) = u'(3)\,v(3)+u(3)\,v'(3)$ — and only then are the four numbers imported from the table or the graph. The structure is the point; the arithmetic is the answer. Both are needed, and they are scored separately.

When not to use it. $x^2(x+1)$ is a product, but it is also $x^3 + x^2$, and the power rule on the expanded form is faster and safer. Expand when the factors are polynomials; use the rule when one factor is $\sin x$, e^x, $\ln x$, or anything that will not multiply out.

CHECK 3 — EXPLAIN THE MISSING TERM

Divide the corner by h and group it as $(\Delta u/h)\,\Delta v$. The first factor goes to u', a finite number, and the second goes to 0, so the product goes to 0 — one factor's *rate* times the other factor's *change*, and the change is what dies. No: $u'v'$ is not zero. On $u = v = x$ it is 1 while the derivative of x^2 is $2x$. The corner vanishing and $u'v'$ vanishing are different statements, and only the first one is true; keeping the difference quotient in the explanation is what keeps them apart.

THE QUOTIENT RULE — AND THE SIGN THAT HAS A SIDE

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'\,v - u\,v'}{v^2}.$$

The numerator is the product rule's two terms with a minus between them, and unlike the product rule, the **order matters**: the derivative of the top comes first. Swap them and the sign of the whole derivative flips — an error that survives every later step and produces a tangent line sloping the wrong way. The denominator is squared, and it is the *original* denominator, not its derivative.

The self-check that never fails. Run the rule on $\frac{1}{x}$: $u = 1$, $v = x$, so the derivative is $\frac{0\cdot x - 1\cdot 1}{x^2} = -\frac{1}{x^2}$. The power rule on x^{-1} gives $-x^{-2}$, the same thing. If the quotient rule in your head gives $+\frac{1}{x^2}$ here, the order is backwards and you have found out for free.

When to avoid it. A quotient whose bottom is a single power of x is a sum of powers in disguise: $\frac{x^2+1}{x^3} = x^{-1} + x^{-3}$, and the power rule term by term is shorter than the quotient rule and has no sign to get wrong. Problem 10 does it both ways so you can watch them agree.

TANGENT, COTANGENT, SECANT, COSECANT — FOUR QUOTIENTS, ONE METHOD

Nothing new is memorized here unless you want it to be. Each of the four is a quotient of sine and cosine, and the quotient rule produces its derivative on demand:

$$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

The other three go the same way. The results, for reference:

Function	Written as	Derivative
$\tan x$	$\sin x/\cos x$	$\sec^2 x$
$\cot x$	$\cos x/\sin x$	$-\csc^2 x$
$\sec x$	$1/\cos x$	$\sec x \tan x$
$\csc x$	$1/\sin x$	$-\csc x \cot x$

The pattern is worth one sentence: every *co*-function's derivative carries a minus sign (cos, cot, csc), and every derivative comes in the same family it started in. A student who remembers that sentence can rebuild the table from $\tan x \to \sec^2 x$ in a minute; a student who remembers only the table has four signs to lose.

PROBLEM 1

Average, then instantaneous: a rate the words asked for

A ball is thrown straight up and its height in feet after t seconds is $h(t) = -16t^2 + 64t$. *(a) Find the average velocity over* $[1, 2]$*, over* $[1, 1.5]$*, and over* $[1, 1.1]$*, with units. (b) Using the definition of the derivative, find the instantaneous velocity at* $t = 1$. *(c) Explain in a sentence what* $h'(1)$ *means in this situation.*

BEFORE YOU COMPUTE

Rung 1 of the tree: each part of (a) names an *interval*, so each is an average rate — a difference quotient with both endpoints, and no limit anywhere. Part (b) names an *instant*, and adds "using the definition," which forbids every rule you know: the answer is a limit, it will substitute to $\frac{0}{0}$, and Unit 1's algebra finishes it. Part (c) is the sentence the exam grades; decide now that it will have units in it.

Two habits before the arithmetic. Compute $h(1)$ once and keep it — every part reuses it. And set up the difference quotient with the function's name in it before any numbers go in; that line is what the CED calls the structure, and it is scored on its own.

WORKING

$h(1) = -16 + 64 = 48$ feet, used throughout.

(a) Three secant slopes.

$$\frac{h(2)-h(1)}{2-1} = \frac{64-48}{1} = 16 \text{ ft/s}, \qquad \frac{h(1.5)-h(1)}{1.5-1} = \frac{60-48}{0.5} = 24 \text{ ft/s},$$

$$\frac{h(1.1)-h(1)}{1.1-1} = \frac{51.04-48}{0.1} = 30.4 \text{ ft/s},$$

where $h(2) = -64 + 128 = 64$, $h(1.5) = -36 + 96 = 60$, and $h(1.1) = -19.36 + 70.4 = 51.04$. The intervals are shrinking onto $t = 1$ and the averages are climbing: 16, 24, 30.4. They are heading somewhere, and (b) says where.

(b) The definition, in the h-form — written with a different letter for the increment, since h is the function's name here:

$$h'(1) = \lim_{k\to 0} \frac{h(1+k)-h(1)}{k}.$$

Expand the numerator before touching the limit:

$$h(1+k) = -16(1+k)^2 + 64(1+k) = -16 - 32k - 16k^2 + 64 + 64k = 48 + 32k - 16k^2,$$

so

$$\frac{h(1+k)-h(1)}{k} = \frac{48+32k-16k^2-48}{k} = \frac{32k-16k^2}{k} = \frac{k\,(32-16k)}{k} = 32-16k \qquad (k \neq 0).$$

Substituting $k = 0$ at the start would have given $\frac{0}{0}$; canceling the k first gives $32 - 16k$, and now the limit is a substitution:

$$h'(1) = \lim_{k\to 0}(32-16k) = 32 \text{ ft/s}.$$

The averages in (a) were 16, 24, 30.4 — and $32 - 16k$ at $k = 1, 0.5, 0.1$ is exactly 16, 24, 30.4. Part (a) was the limit being approached from the outside; (b) is where it arrives.

(c) The sentence: *At* $t = 1$ *second, the ball is rising at* 32 *feet per second.* Three things in it — the moment, the rate, the units — and "rising" because the sign is positive.

Sense check. The rules of the next cards give $h'(t) = -32t + 64$, so $h'(1) = 32$. The definition and the rule agree, which is the only relationship they are allowed to have.

ANSWER

(a) 16, 24, 30.4 ft/s (b) $h'(1) = 32$ ft/s (c) at $t = 1$ s the ball is rising at 32 ft/s

WATCH OUT

The most expensive error in (b) is not algebraic: it is answering with a rule. "Using the definition" means the limit must appear on the page, with the $\frac{0}{0}$ resolved by cancellation; $h'(t) = -32t + 64$ written down cold earns nothing here, however right it is. The second is substituting $k = 0$ into the quotient before canceling and stopping at "undefined." Both words are in play and they are not the same word. The quotient *at* $k = 0$ genuinely is undefined — you may not divide by zero. What is not undefined is the limit: $\frac{0}{0}$ is an *indeterminate form*, which says the limit is not yet determined, not that it fails to exist. Unit 1 spent a whole card on that distinction, and this is where it gets paid.

In (a), a velocity is a signed quantity and a unit is part of the answer. "16" is a number; "16 ft/s" is an average velocity. And in (c), the sentence has to name the instant: "the ball is rising at 32 ft/s" is true only at $t = 1$, and the grader is looking for the $t = 1$.

CONNECTION

The three averages in (a) are the secant slopes in the first toolbox card, and 32 is the tangent's. Unit 4 will ask the reverse question — given h', when is the ball at its highest? — and the answer is where $h'(t) = 0$: $t = 2$. Notice that (a)'s average over $[1, 2]$ was 16 while the instantaneous rate at 1 is 32: the ball is slowing all the way up, so the average over the interval is less than the rate at its start. That comparison — average versus instantaneous, and which is bigger — is a Unit 5 question about concavity, arriving early.

ABOUT THIS EXCERPT

This is the opening of a 39-page guide: the diagnostic tree, the full Master Toolbox, and the first worked problem. 10 more problems follow in the complete guide, each worked the same way — what to notice before you start, every step shown, and the mistake that problem invites. The complete guide is shared with families during the fit conversation.

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